Exam Analysis 2014

Date	:	03-11-2014
Time	:	14.00 - 17.00
Place	:	Aletta Jacobshal

Provide clear arguments for all your answers; 'yes' or 'no' answers are not allowed. In your argumentation you may use all theorems and statements in the book. However, you should indicate which theorems/ statements you are using.

The detailed grading scheme can be found below.

1. Let $A \subset \mathbb{R}$ be bounded, and let L be the set of limit points of A.

   (a) Prove that L is bounded.

   (b) Assume L is non-empty. Let $c = \sup L$. Prove that c is a limit point of A.

   (c) Prove that for all $\epsilon > 0$ there are only finitely many elements $x \in A$ with $x > c+\epsilon$, while there are infinitely many elements $x \in A$ with $x > c - \epsilon$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

   $$f(tx) = tf(x)$$

   for all $x, t \in \mathbb{R}$.

   (a) Show that f is continuous everywhere.
   (**Hint:** for any two points x, y with $x \neq 0$ we can find $t \in \mathbb{R}$ such that $y = tx$.)

   (b) Show that f is differentiable everywhere, and prove that

   $$xf'(x) = f(x)$$

   for all x.

3. Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions, which are differentiable on (a, b). Assume that

   $$f(a) \leq g(a), \quad f'(x) < g'(x),\ x \in (a, b)$$

   Prove that

   $$f(x) < g(x),$$

   for all $x \in (a, b]$.

4. Let $f_n : [a, b] \to [c, d]$, and $g_n : [c, d] \to \mathbb{R}$, $n = 1, 2 \cdots, \infty$, be sequences of continuous functions. Let f_n converge uniformly to f, and let g_n converge uniformly to g.
   Prove that the sequence of functions $h_n : [a, b] \to \mathbb{R}$, defined as the composition

   $$h_n := g_n \circ f_n, \quad n = 1, 2 \cdots, \infty$$

   converges uniformly to the function $h := g \circ f$.

5. Consider the series

   $$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

   (a) Prove that the series converges uniformly on all intervals $[a, \infty)$ with $a > 0$.

   (b) Show that the function is differentiable on all those intervals.

6. (a) Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable, and assume that $\int_0^1 f = 2$. Prove that there exists an $x \in (0, 1)$ such that

   $$\int_0^x f = 1$$

   (b) Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$$

   Is f Riemann integrable on $[0, 1]$?

Grading scheme:

Total 100, Free 10.

1. a: 5, b: 5, c: 5.
2. a: 8, b: 7.
3. 15.
4. 15.
5. a: 10, b: 5.
6. a: 9, b: 6.